Worcester County Mathematics League

Varsity Meet 1 - October 2, 2019

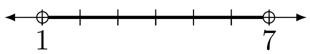
COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Answer Key

Round 1 - Arithmetic

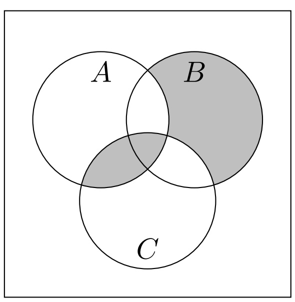
1. 112
2. $-\dfrac{78}{5}$ (note: no mixed fractions/decimals)
3. $\dfrac{1}{11}$ or $0.\overline{09}$

Round 2 - Algebra I

1. 

(note: open circles)
2. \$0.25 or 25¢ [note: .25¢ or “25” are incorrect]
3. 11 students [not including Cathy]

Round 3 - Set Theory

1. $\{o, r, a\}$ (any order, braces optional)
2. 

3. $\dfrac{1}{2}$ or 0.5

Round 4 - Measurement

1. 30 inches
2. $\dfrac{1}{3}$ or one-third or $0.\overline{3}$
3. $\dfrac{1}{2}$ [2 is not accepted]

Round 5 - Polynomial Equations

1. $x = 5, -3$
2. $2i$
3. $d = -\dfrac{1}{3}$ [note negative value]

Team Round

1. -45
2. $a = 10$
3. 22 students
4. 500π square feet
5. $\dfrac{2}{3}$
6. 34560
7. -4 or $-\dfrac{7}{2}$
8. $x = 2, y = -4$ (note: $(2, -4)$ is acceptable)
9. $80°$

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate the expression
$$\left[-2(4-6)^2\right]^2 - 4\left[-3(6 \div 3)^2\right]$$

2. If $x = \dfrac{2}{3}$ and $y = \dfrac{3}{4}$, evaluate
$$\frac{x + \frac{1}{x}}{y - \frac{x}{y}}$$
and express your answer as an improper fraction in lowest terms.

3. What is the positive difference between $A = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1}}}}$ and $B = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \ddots}}}$?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Shepherd Hill, Doherty, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Graph the solution set for
$$11 - 2x > -3 \qquad \text{and} \qquad 7 - 3x < 4.$$

2. Martha and Mike each head to the grocery store to get some candy. Martha buys two bags of M&Ms and three bags of Skittles for a total of \$4.40, while Mike buys eight bags of M&Ms and four bags of Skittles for a total of \$10.40. How much change would you receive if you walked into the same store to buy one bag of each type of candy and paid the cashier \$2.00?

3. It's first period and Cathy is watching many of her classmates struggling to stay awake in Mr. Bueller's English class. She notices that one-third of the class fell asleep after ten minutes of class. Five minutes later, she sees one-fourth of those who were still awake dozing off. Ten minutes after that, she watches three more students begin to snore. At the end of class, Mr. Bueller finally catches on and drops a book loudly on the floor, waking up two-thirds of the students who were just sleeping. If six students were still asleep at that time, how many of Cathy's classmates stayed awake through the entire class?

ANSWERS

(1 pt) 1. $\longleftrightarrow$

(2 pts) 2. ______________________

(3 pts) 3. ______________________ students

St. Peter-Marian, QSC, Shepherd Hill/QSC

Worcester County Mathematics League

Varsity Meet 1 - October 2, 2019

Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.

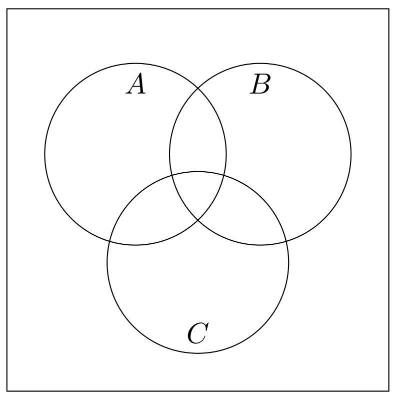
NO CALCULATORS ALLOWED

1. Set $A = \{e, q, u, a, l\}$, set $B = \{w, o, r, k\}$, set $C = \{f, o, r\}$, and set $D = \{p, a, y\}$. Determine the elements of

$$(A \cup B) \cap C \cup (A \cap D).$$

2. If $A \setminus B$ is the set of all elements in A and not in B (known as a the set difference, or relative complement), shade the area of the venn diagram below to reflect the set

$$(A \cap C) \cup (B \setminus (A \cup C)).$$



3. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If three distinct elements of S are chosen at random, what is the probability that the sum of those elements is divisible by 2?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. (shade diagram above)

(3 pts) 3. ______________________

QSC, Bartlett/QSC, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Round 4 - Measurement

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. A rectangular piece of sheet metal is 3 inches longer than it is wide. If the length and width are both increased by 2 inches, the area increases by 34 square inches. What is the perimeter of the original sheet of metal?

2. Betty's ice cream is melting! She received her perfect sphere of ice cream in a perfect cylindrical cup with a bottom and sides but no top. The fit was perfect; the sphere of ice cream was tangent to the base of the cup as well as all the sides, and the cup's height was exactly half of the cylinder's diameter. Due to the hot summer sun, her ice cream is melting and filling the cup. What fraction of the ice cream must she eat so the rest of it (uneaten) will melt into the cup?

3. A regular hexagon is inscribed in a circle, which is itself inscribed in a regular triangle. Determine the ratio of the area of the hexagon to the area of the triangle in simplest form.

ANSWERS

(1 pt) 1. ____________________ inches

(2 pts) 2. ____________________

(3 pts) 3. ____________________ Southbridge, QSC, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find all values of x such that
$$(x-4)(x+2)=7.$$

2. Simplify the expression $(i^4+i^3+i^2-1)^2$.

3. The polynomial equation
$$3x^2+14x+16=d$$
has a double root. Solve for d.

ANSWERS

(1 pt) 1. $x=$ ____________________

(2 pts) 2. ____________________

(3 pts) 3. $d=$ ____________________

Bartlett, Bartlett, Algonquin

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $a\&b = 3a + 4b$, evaluate $c\&d$ for

$$c = 2 + 3 \times 4 - 9 \qquad \text{and} \qquad d = 1 - 4 \div 2 - 7 \times 2.$$

2. Find the positive integer a satisfying the equation below.

$$\sqrt{\frac{4^{20} - 2^{21} + 1}{2^{20} + 2^{11} + 1}} = 2^a - 1$$

3. Thirty-five students were asked to try two new sodas labeled A and B to decide if they liked either, both, or neither. Twice as many students liked B as liked A, and half the number who liked A liked neither. If 7 students reported liking both sodas, how many only liked one type of soda?

4. A dog is tied to a 25-foot-long lead attached to one corner of a regular hexagonal building in the middle of a large, flat field. The sides of the building are of length ten feet. What is the area of the field accessible to the dog without breaking the lead? Express your answer as a multiple of π.

5. Let c be a rational number. One root of $3x^2 - 7x + c = 0$ is $x = \frac{1}{3}$. Find the product of both roots.

6. Consider all sets of six integers that have a mean, median, unique mode, and range of 6. Find the set with the largest possible integer; what is the product of the data?

7. Let $g(2x - 1) = 8x^2 - 2x$. Find the roots of $g(x + 3)$.

8. Solve for x and y:

$$\begin{cases} x(y-4) = y(x+2) \\ \dfrac{3x - y - 1}{3} = \dfrac{x+4}{2} \end{cases}$$

9. Points A, B, C, D, and E lie consecutively along a circle with center O. If $\overline{AD}$ is a diameter of the circle, $\overline{CO} \cong \overline{AE}$, and $\measuredangle ABC = 130°$, find the measure of $\angle CDE$.

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Team Round Answer Sheet

ANSWERS

1. ____________________________

2. $a =$ ____________________________

3. ____________________________ students

4. ____________________________ square feet

5. ____________________________

6. ____________________________

7. ____________________________

8. $x =$ ______________ $y =$ ______________

9. ____________________________ degrees

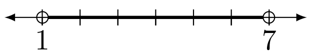
Oxford, Sutton, QSC, Bancroft, Groton-Dunstable, QSC, QSC, West Boylston, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 2, 2019
Answer Key

Round 1 - Arithmetic

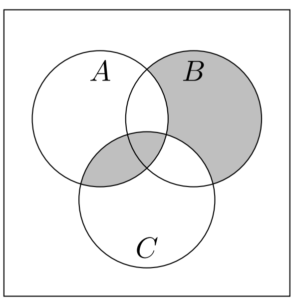
1. 112
2. $-\dfrac{78}{5}$ (note: no mixed fractions/decimals)
3. $\dfrac{1}{11}$ or $0.\overline{09}$

Round 2 - Algebra I

1. 

(note: open circles)
2. \$0.25 or 25¢ [note: .25¢ or "25" are incorrect]
3. 11 students [not including Cathy]

Round 3 - Set Theory

1. $\{o, r, a\}$ (any order, braces optional)
2. 

3. $\dfrac{1}{2}$ or 0.5

Round 4 - Measurement

1. 30 inches
2. $\dfrac{1}{3}$ or one-third or $0.\overline{3}$
3. $\dfrac{1}{2}$ [2 is not accepted]

Round 5 - Polynomial Equations

1. $x = 5, -3$
2. $2i$
3. $d = -\dfrac{1}{3}$ [note negative value]

Team Round

1. -45
2. $a = 10$
3. 22 students
4. 500π square feet
5. $\dfrac{2}{3}$
6. 34560
7. -4 or $-\dfrac{7}{2}$
8. $x = 2, y = -4$ (note: $(2, -4)$ is acceptable)
9. $80°$

Round 1 - Arithmetic

1. Evaluate the expression

$$\left[-2(4-6)^2\right]^2 - 4\left[-3(6\div 3)^2\right]$$

Solution: Simplify using order of operations.

$$\left[-2(4-6)^2\right]^2 - 4\left[-3(6\div 3)^2\right] = \left[-2(-2)^2\right]^2 - 4\left[-3(2)^2\right]$$
$$= \left[-2(4)\right]^2 - 4\left[-3(4)\right]$$
$$= \left[-8\right]^2 - 4\left[-12\right]$$
$$= 64 + 48 = \boxed{112}$$

2. If $x = \dfrac{2}{3}$ and $y = \dfrac{3}{4}$, evaluate

$$\frac{x+\frac{1}{x}}{y-\frac{x}{y}}$$

and express your answer as an improper fraction in lowest terms.

Solution: Substitute and then simplify.

$$\frac{x+\frac{1}{x}}{y-\frac{x}{y}} = \frac{\frac{2}{3}+\frac{1}{\frac{2}{3}}}{\frac{3}{4}-\frac{\frac{2}{3}}{\frac{3}{4}}} = \frac{\frac{2}{3}+\frac{3}{2}}{\frac{3}{4}-\frac{8}{9}} = \frac{\frac{13}{6}}{-\frac{5}{36}} = \boxed{-\frac{78}{5}}$$

3. What is the positive difference between $A = 1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}}$ and $B = 1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\ddots}}}$?

Solution: Starting with A, we evaluate

$$A = 1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}} = 1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{3}}} = 1+\cfrac{2}{1+\cfrac{6}{5}} = 1+\frac{10}{11} = \frac{21}{11}$$

Then we determine B. Note that $B = 1+\dfrac{2}{B}$ so $B^2 = B+2$ and

$$0 = B^2 - B - 2 = (B-2)(B+1)$$

Since B must be positive, our only value for B must be 2 and $|A-B| = \left|\dfrac{21}{11}-2\right| = \left|\dfrac{-1}{11}\right| = \boxed{\dfrac{1}{11}}$.

Round 2 - Algebra I

1. Graph the solution set for

$$11 - 2x > -3 \qquad \text{and} \qquad 7 - 3x < 4.$$

Solution: The first inequality can be simplified to

$$11 - 2x > -3$$

$$-2x > -14$$

$$x < 7$$

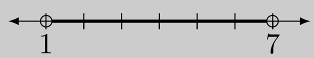
and the second can be simplified to

$$7 - 3x < 4$$

$$-3x < -3$$

$$x > 1.$$

Therefore, the only numbers that satisfy both of these inequalities are numbers x such that $1 < x < 7$ and is graphed as:



2. Martha and Mike each head to the grocery store to get some candy. Martha buys two bags of M&Ms and three bags of Skittles for a total of \$4.40, while Mike buys eight bags of M&Ms and four bags of Skittles for a total of \$10.40. How much change would you receive if you walked into the same store to buy one bag of each type of candy and paid the cashier \$2.00?

Solution: Let M represent bags of M&Ms and S represent bags of Skittles. We know two equations:

$$2M + 3S = \$4.40$$

$$8M + 4S = \$10.40$$

Dividing the second equation by 4, we arrive at the system

$$\begin{cases} 2M + 3S = \$4.40 \\ 2M + S = \$2.60 \end{cases}$$

Subtracting the second equation from the first, we find that $2S = \$1.80$ and that a bag of Skittles costs \$0.90. This means that 2 bags of M&Ms cost \$1.70, or \$0.85 each. Combined, they would cost \$1.75, and you'd receive $\boxed{\text{25 cents or \$0.25 or 25¢}}$ in change when paying \$2.00.

(Alternatively, adding the equations together gives us $4M + 4S = \$7.00$, which, when divided by 4, gives us $M + S = \$1.75$. Slick, huh? Candy tastes better when the journey is satisfying.)

3. It's first period and Cathy is watching many of her classmates struggling to stay awake in Mr. Bueller's English class. She notices that one-third of the class fell asleep after ten minutes of class. Five minutes later,

she sees one-fourth of those who were still awake dozing off. Ten minutes after that, she watches three more students begin to snore. At the end of class, Mr. Bueller finally catches on and drops a book loudly on the floor, waking up two-thirds of the students who were just sleeping. If six students were still asleep at that time, how many of Cathy's classmates stayed awake through the entire class?

Solution: Let n be the total number of students in the classroom. When one-third of the class falls asleep, two-thirds of the students, or $\frac{2}{3}n$ students, are still **awake**. When one-fourth of those remaining students fall asleep, $\frac{3}{4} \cdot \frac{2}{3}n = \frac{1}{2}n$ are still **awake**. Note that this is half the class, so $\frac{1}{2}n$ students at this time that are also **asleep**.

When three more students fall asleep, $\frac{1}{2}n + 3$ are now **asleep**. When two-thirds wake up, one-third of those students are still asleep, or $\frac{1}{6}n + 1$ students. This represents six students, so

$$\frac{1}{6}n + 1 = 6$$

$$n + 6 = 36$$

$$n = 30$$

means that there are thirty total students in the classroom.

At the peak of sleepytime, $\frac{1}{2} \cdot 30 + 3 = 18$ students are sleeping. This means twelve students (and $\boxed{11}$ of Cathy's classmates) stayed awake the whole class.

Round 3 - Set Theory

1. Set $A = \{e, q, u, a, l\}$, set $B = \{w, o, r, k\}$, set $C = \{f, o, r\}$, and set $D = \{p, a, y\}$. Determine the elements of

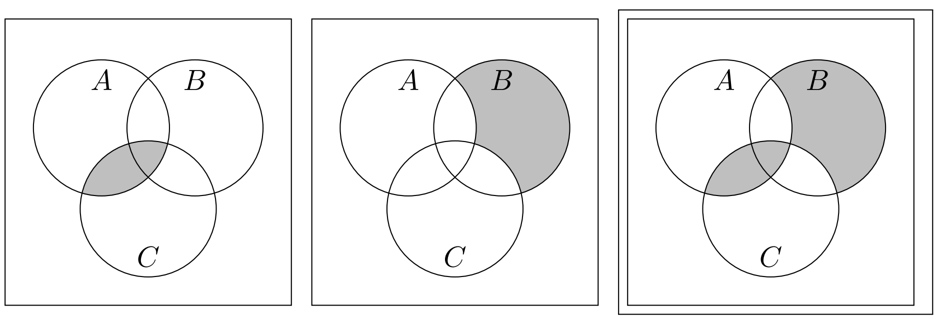
$$(A \cup B) \cap C \cup (A \cap D).$$

Solution: The elements in $A \cup B$ are all the elements in A or B, or $\{e, q, u, a, l, w, o, r, k\}$. The elements in $(A \cup B) \cap C$ are all elements in $A \cup B$ and C: $\{o, r\}$. The elements in $A \cap D$ are the elements in both sets: $\{a\}$. The union of this set and the set $(A \cup B) \cap C$ is $\boxed{\{o, r, a\}}$ (note: order of elements does not matter).

2. If $A \setminus B$ is the set of all elements in A and not in B (known as a the set difference, or relative complement), shade the area of the venn diagram below to reflect the set

$$(A \cap C) \cup (B \setminus (A \cup C)).$$

Solution: Starting with the first set, $A \cap C$, we are looking at all elements in A as well as C, which is the overlapping area between sets A and C [see left diagram]. The next set can be read as "the elements of B that are not in sets A or C", meaning the elements that are only in B and no other set [see middle diagram]. The union of these two sets is the collection of the elements in both of the described sets [see right diagram].



3. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If three distinct elements of S are chosen at random, what is the probability that the sum of those elements is divisible by 2?

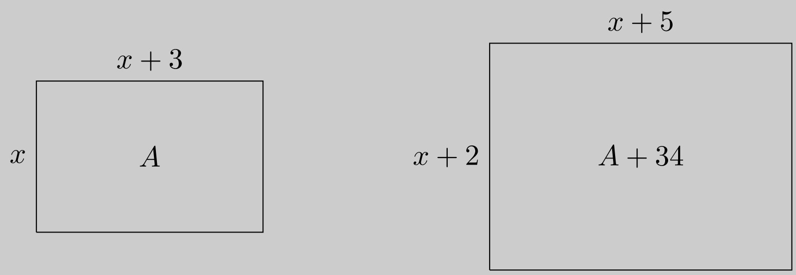
Solution: There are four possibilities when choosing 3 numbers from the set. Either all three are even (EEE), all three are odd (OOO), two are even with one being odd (EEO), or two are odd with one being even (OOE). The two situations where the sum will be even are EEE and OOE, so let's focus on those.

Overall, there are $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$ ways for us to pick three balls at random from a group of ten. Since there are five odd numbers in S and five even numbers in S. The likelihood of choosing, three even numbers in a row at random is $\frac{5}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{10}{120}$. The likelihood of choosing two odd numbers and one even number, which sums to an even number, is $\frac{5}{10} \cdot \frac{4}{9} \cdot \frac{5}{8} = \frac{50}{120}$. Together, the likelihood of choosing three numbers whose sum is divisible by two is $\frac{10}{120} + \frac{50}{120} = \frac{60}{120} = \boxed{\frac{1}{2}}$. **Alternate Solution:** Since the chance of choosing EEE is identical to choosing OOO, and the chance of choosing OOE is identical to choosing EEO, then the chance of having an even sum is identical to the chances of having an odd sum. Hence, $\frac{1}{2}$.

Round 4 - Measurement

1. A rectangular piece of sheet metal is 3 inches longer than it is wide. If the length and width are both increased by 2 inches, the area increases by 34 square inches. What is the perimeter of the original sheet of metal?

Solution: The original piece of sheet metal has dimensions x by $x+3$, meaning the larger piece of sheet metal has dimensions $x+2$ and $x+5$.



Since the area of the larger sheet is 34 units larger than the first,

$$x(x+3)+34=(x+2)(x+5)$$

$$x^2+3x+34=x^2+7x+10$$

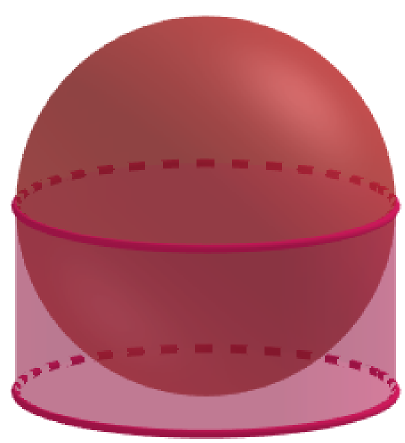
$$24=4x$$

$$6=x$$

and the perimeter of the original sheet of metal is $2\cdot(6+9)=\boxed{30 \text{ inches}}$.

2. Betty's ice cream is melting! She received her perfect sphere of ice cream in a perfect cylindrical cup with a bottom and sides but no top. The fit was perfect; the sphere of ice cream was tangent to the base of the cup as well as all the sides, and the cup's height was exactly half of the cylinder's diameter. Due to the hot summer sun, her ice cream is melting and filling the cup. What fraction of the ice cream must she eat so the rest of it (uneaten) will melt into the cup?

Solution: Let r be the radius of the sphere of ice cream as well as the radius and the height of the cup that the ice cream is in. The volume of the sphere is $\frac{4}{3}\pi r^3$ whereas the volume of the cylinder is $B\cdot h=\pi r^2\cdot r=\pi r^3$. Since the sphere is four-thirds of the voulme of the cylinder, which means Betty needs to eat $\boxed{\frac{1}{3}}$ of her ice cream so the rest melts into the cup.

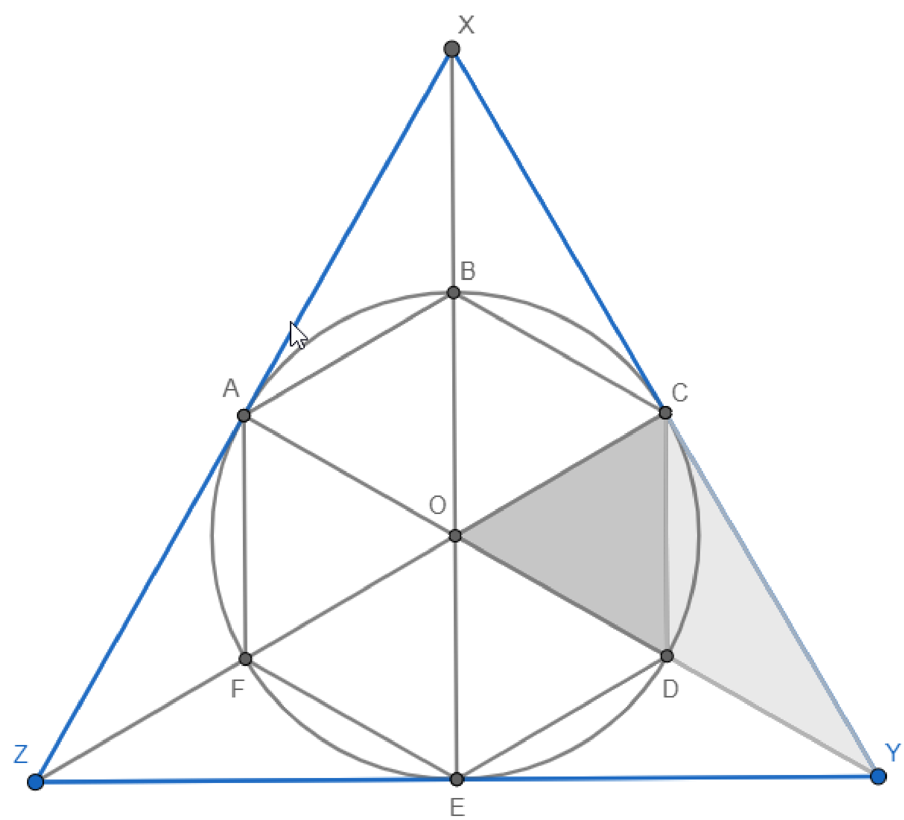


3. A regular hexagon is inscribed in a circle, which is itself inscribed in a regular triangle. Determine the ratio of the area of the hexagon to the area of the triangle in simplest form.

Solution: Let r be the radius of the circle. The hexagon is comprised of six equilateral triangles with each side being length r. The area of each of these triangles is $\frac{r^2\sqrt{3}}{4}$, meaning the area of the hexagon is $\frac{6r^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{2}$.

To determine the area of the large triangle (and referencing the diagram below) we know that ΔOCX is a $30° - 60° - 90°$ right triangle with short leg r, which means $XC = r\sqrt{3}$ and the side length overall is $2r\sqrt{3}$. The area of this triangle is $\frac{(2r\sqrt{3})^2\sqrt{3}}{4} = \frac{12r^2\sqrt{3}}{4} = \frac{6r^2\sqrt{3}}{2}$. This is twice the area of the hexagon, so the radio of the area of the hexagon to the area of the triangle is $\boxed{\frac{1}{2}}$.

Alternate solution: Point O in the diagram is the centroid of ΔXYZ, and that point marks two-thirds of each median's length from the vertex to the opposite side. Hence, since $AO = r$, so do OD and DY. This means that the area of ΔOCD is equal to the area of ΔYCD (same base length, same height), so each of the six equilateral triangles contained in hexagon $ABCDEF$ is matched by an equal area triangle outside of it. Therefore, the area of the hexagon is one-half the area of the larger triangle.



Round 5 - Polynomial Equations

1. Find all values of x such that
$$(x-4)(x+2)=7.$$

Solution:
$$(x-4)(x+2)=7$$
$$x^2-2x-8=7$$
$$x^2-2x-15=0$$
$$(x-5)(x+3)=0$$
Therefore, $\boxed{x=-3 \text{ or } 5}$.

2. Simplify the expression $(i^4+i^3+i^2-1)^2$.

Solution:
$$(i^4+i^3+i^2-1)^2$$
$$=(1+(-i)+(-1)-1)^2$$
$$=(-i-1)^2$$
$$=i^2+2i+1$$
$$=-1+2i+1$$
$$=\boxed{2\text{i}}$$

3. The polynomial equation
$$3x^2+14x+16=d$$
has a double root. Solve for d.

Solution: Adjusting the equation above gives us $3x^2+14x+16-d=0$ and dividing through by 3 to let the leading coefficient be equal to 1 gives us
$$x^2+\frac{7}{3}x+\frac{16-d}{3}=0.$$
In order for the polynomial to have a double root, the left-hand side must be a perfect square trinomial where the constant term $\frac{16-d}{3}$ is the square of half the linear coefficient $\frac{14}{3}$.
$$\left(\frac{14}{3}\cdot\frac{1}{2}\right)^2=\frac{49}{9}$$
Now we set and solve for d.
$$\frac{49}{9}=\frac{16-d}{3}$$

$$\frac{49}{9} = \frac{48 - 3d}{9}$$

$$49 = 48 - 3d$$

$$\boxed{d = -\frac{1}{3}}$$

Alernate Solution: A quadratic polynomial has a double root when the discriminant is equal to 0. This means

$$b^2 - 4ac = 0$$

$$196 - 4(3)(16 - d) = 0$$

$$196 - 192 + 12d = 0$$

$$12d = -4$$

$$d = -\frac{1}{3}$$

Team Round

1. If $a\&b = 3a + 4b$, evaluate $c\&d$ for

$$c = 2 + 3 \times 4 - 9 \qquad \text{and} \qquad d = 1 - 4 \div 2 - 7 \times 2.$$

Solution: Simplifying for c and d:

$$\begin{aligned} c &= 2 + 3 \times 4 - 9 \\ c &= 2 + 12 - 9 \\ c &= 5 \end{aligned} \qquad\qquad \begin{aligned} d &= 1 - 4 \div 2 - 7 \times 2 \\ d &= 1 - 2 - 14 \\ d &= -15 \end{aligned}$$

So $c\&d = 3c + 4d = 3(5) + 4(-15) = 15 - 60 = \boxed{-45}$.

2. Find the positive integer a satisfying the equation below.

$$\sqrt{\frac{4^{20} - 2^{21} + 1}{2^{20} + 2^{11} + 1}} = 2^a - 1$$

Solution: The numerator and denominator of the radicand can be factored:

$$\begin{aligned} 4^{20} - 2^{21} + 1 &= \left(2^2\right)^{20} - 2^{21} + 1 \\ &= \left(2^{20}\right)^2 - 2 \cdot 2^{20} + 1 \\ &= \left(2^{20} - 1\right)^2 \end{aligned} \qquad\qquad \begin{aligned} 2^{20} + 2^{11} + 1 &= \left(2^{10}\right)^2 + 2^{11} + 1 \\ &= \left(2^{10}\right)^2 + 2 \cdot 2^{10} + 1 \\ &= \left(2^{10} + 1\right)^2 \end{aligned}$$

This can help simplify the left-side expression to

$$\sqrt{\frac{4^{20} - 2^{21} + 1}{2^{20} + 2^{11} + 1}} = \sqrt{\frac{(2^{20} - 1)^2}{(2^{10} + 1)^2}} = \frac{(2^{20} - 1)}{(2^{10} + 1)} = \frac{\left(\left(2^{10}\right)^2 - 1\right)}{(2^{10} + 1)} = \frac{(2^{10} + 1)(2^{10} - 1)}{(2^{10} - 1)} = 2^{10} - 1.$$

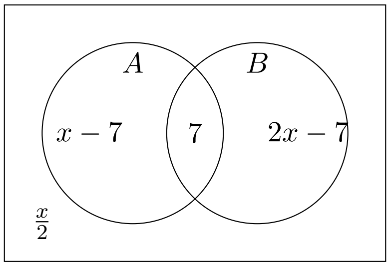
Comparing this to the expression $2^a - 1$, we see that $\boxed{a = 10}$.

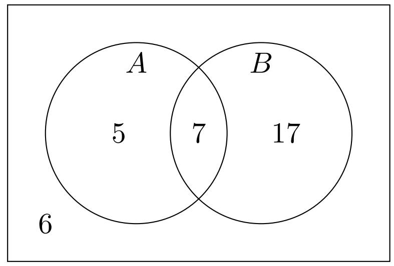
3. Thirty-five students were asked to try two new sodas labeled A and B to decide if they liked either, both, or neither. Twice as many students liked B as liked A, and half the number who liked A liked neither. If 7 students reported liking both sodas, how many only liked one type of soda?

Solution: Let x be the number of students who reported liking soda A. This means that $2x$ is the number of students who reported liking B. Since 7 students reported liking both and $\frac{x}{2}$ students liked neither, we can create an equation that sets all four disjoint sets (neither, only A, only B, both A and B) equal to the total number of students interviewed.

$$(x - 7) + 7 + (2x - 7) + \frac{x}{2} = \underline{3.5x - 7 = 35}$$

Solving for x we find $3.5x = 42 \longrightarrow x = 12$. We can fill out our diagram and see that a total of $\boxed{22}$ students reported liking only one of the two sodas.

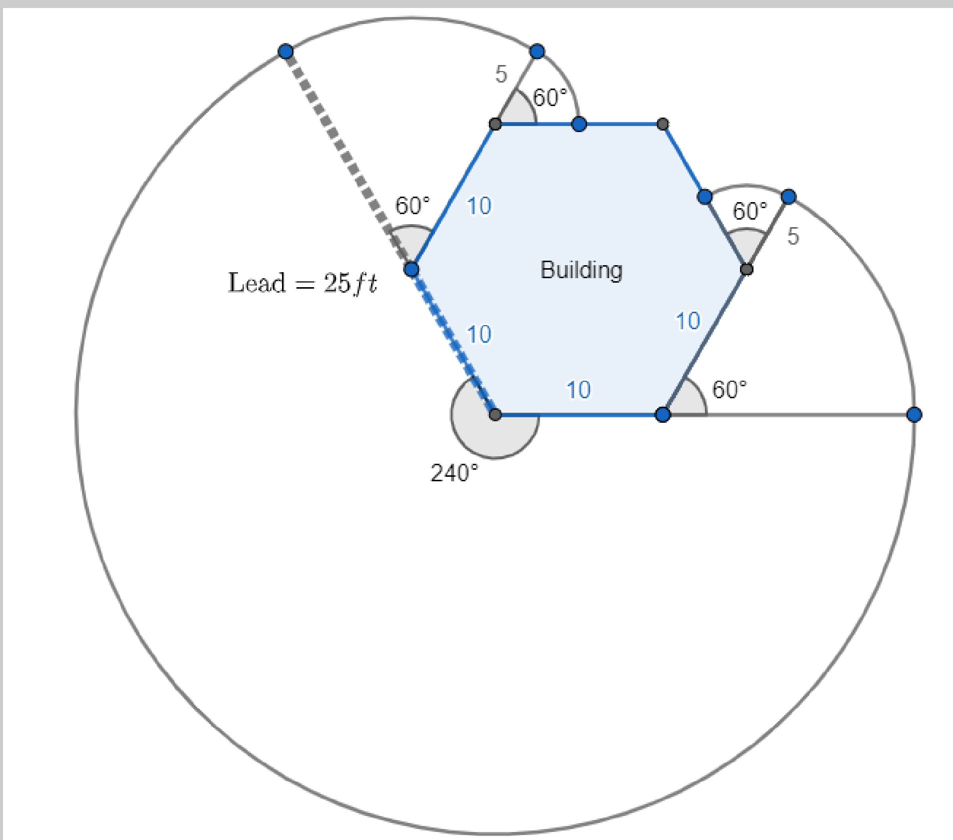




4. A dog is tied to a 25-foot-long lead attached to one corner of a regular hexagonal building in the middle of a large, flat field. The sides of the building are of length ten feet. What is the area of the field accessible to the dog without breaking the lead? Express your answer as a multiple of π.

Solution: The dog being attached to one of the corners allows it to roam in the areas detailed below. Note that if the dog is stretching the lead to the full 25 feet, once it tries to move around the building the length of the lead is effectively cut by 10 feet each corner. This means that the area the dog can roam around is two-thirds of a 25-foot radius circle plus one-third of a 15-foot radius circle plus one-third of a 5-foot radius circle.

$$\frac{2}{3}625\pi + \frac{1}{3}225\pi + \frac{1}{3}25\pi = \frac{1500}{3}\pi = \boxed{500\pi \text{ square feet}}$$



5. Let c be a rational number. One root of $3x^2 - 7x + c = 0$ is $x = \frac{1}{3}$. Find the product of both roots.

Solution: By the remainder theorem, if $f(x)$ is a polynomial and $f(a) = 0$, then a is a root of the polynomial. Letting $f(x) = 3x^2 - 7x + c$, and knowing $\dfrac{1}{3}$ is a root of f, we find that

$$f\left(\frac{1}{3}\right) = \frac{3}{9} - \frac{7}{3} + c = \frac{1}{3} - \frac{7}{3} + c = -2 + c = 0$$

so c must be equal to 2. We now can replace c with 2 and solve the quadratic.

$$3x^2 - 7x + 2 = 0$$

$$(3x - 1)(x - 2) = 0$$

$$x = \frac{1}{3} \text{ or } 2.$$

The product of the roots is $\boxed{\dfrac{2}{3}}$.

Alternate Solution: Synthetically divide the quadratic by $\left(x - \frac{1}{3}\right)$:

$$\begin{array}{c|ccc} & 3 & -7 & c \\ \frac{1}{3} & & 1 & -2 \\ \hline & 3 & -6 & (-2 + 1c) \end{array}$$

Note that the result is that $3x - 6$ is the quotient with the remainder, which we know to be zero, equal to $-2 + c$. This leads us to determine $c = 2$ and the second root is 2 from the quotient.

Alternate Solution: Let

$$3x^2 - 7x + c = (3x - 1)(x - r)$$

where r is the second root. We see that

$$3x^2 - 7x + c = (3x - 1)(x - r)$$

$$3x^2 - 7x + c = 3x^2 - x - 3rx + r$$

$$3x^2 - 7x + c = 3x^2 - (1 + 3r)x + r$$

Knowing $1 + 3r = 7$ we know $r = 2$ (and $c = 2$ as well).

6. Consider all sets of six integers that have a mean, median, unique mode, and range of 6. Find the set with the largest possible integer; what is the product of the data?

Solution: For a set of data to have a mode of 6, the number 6 must be in the range of the data. Since the range is also 6, the only possible maximum and minimum combinations are 0 and 6, 1 and 7, 2 and 8, 3 and 9, 4 and 10, 5 and 11, or 6 and 12. For the mean to be 6, there must be data above and below 6, so this eliminates the 0/6 and 6/12 ranges.

For the median to be 6 with an even number of data, and to maintain the mode also being 6 (and therefore

actually being a piece of data), the two middle data must both be 6. Let try some cases.
Case 1: the min/max are 5/11. Our set needs to satisfy

$$\{5, a, 6, 6, b, 11\}$$

but the sum of the six data must be 36 to enable our mean to be six. Since $5 + 6 + 6 + 11 = 28$, $a + b$ must equal 8 and this is not possible to do with two numbers between 5 and 11.
Case 2: the min/max are 4/10. Our set needs to satisfy

$$\{4, a, 6, 6, b, 10\}$$

and by the previous logic $a + b$ must equal 10. If $a = b = 5$ then the set is bimodal (not allowed). If $a = 4$ and $b = 6$, though, we maintain a mean, medium, unique mode, and range of 6:

$$\{4, 4, 6, 6, 6, 10\}$$

The product of this data: $(4)^2(6)^3 \cdot 10 = \boxed{34560}$.

7. Let $g(2x - 1) = 8x^2 - 2x$. Find the roots of $g(x + 3)$.

Solution: It appears that $g(t)$ is likely a polynomial function since composing it with a polynomial results in another polynomial. There needs to be an $8x^2$ in $g(2x-1)$, so $g(t)$ must involve squaring the input. If we try

$$g(t) = 2t^2$$

we end up with

$$g(2x - 1) = 2(2x - 1)^2 = 8x^2 - 8x + 2$$

which gets us our first term. We have $-8x$ as well which we'd like to be $-2x$; let's add 3 of the inputs $(2x - 1)$.

$$g(t) = 2t^2 + 3t$$

$$g(2x - 1) = 8x^2 - 8x + 2 + 3(2x - 1) = 8x^2 - 2x - 1$$

We're one away, so let's add 1:

$$g(t) = 2t^2 + 3t + 1$$

$$g(2x - 1) = 8x^2 - 2x - 1 + 1 = 8x^2 - 2x$$

So $g(t) = 2t^2 + 3t + 1$. Another way to get here is to set $2x - 1 = t$, realize that $x = \frac{t+1}{2}$, and then substitute:

$$\begin{aligned} g(t) &= g(2x - 1) \\ &= 8x^2 - 2x \\ &= 8\left(\frac{t+1}{2}\right)^2 - 2\left(\frac{t+1}{2}\right) \\ &= 2(t + 1)^2 - t - 1 \\ &= 2t^2 + 4t + 1 - t - 1 \\ &= 2t^2 + 3t + 1 \end{aligned}$$

Approach #1: Let's find $g(x+3)$ and then its roots.

$$g(x+3) = 2(x+3)^2 + 3(x+3) + 1 = 2x^2 + 12x + 18 + 3x + 9 + 1 = 2x^2 + 15x + 28 = (2x+7)(x+4)$$

The roots of this function are $\boxed{-4 \text{ or } -\frac{7}{2}}$.

Approach #2: Find the roots of $g(t)$ and shift left 3 units.

$$g(t) = 2t^2 + 3t + 1 = (2x+1)(x+1)$$

The roots of this function are -1 and $-\frac{1}{2}$. Shifted left three units, the roots are -4 or $-\frac{7}{2}$.

8. Solve for x and y:

$$\begin{cases} x(y-4) = y(x+2) \\ \dfrac{3x-y-1}{3} = \dfrac{x+4}{2} \end{cases}$$

Solution: Manipulating the first equation, we find

$$\begin{aligned} x(y-4) &= y(x+2) \\ xy - 4x &= xy + 2y \\ -4x &= 2y \\ -2x &= y \end{aligned}$$

Substituting that in the second equation:

$$\begin{aligned} \frac{3x+2x-1}{3} &= \frac{x+4}{2} \\ \frac{5x-1}{3} &= \frac{x+4}{2} \\ 10x - 2 &= 3x + 12 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Thus, $\boxed{x = 2 \text{ and } y = -4}$.

9. Points A, B, C, D, and E lie consecutively along a circle with center O. If $\overline{AD}$ is a diameter of the circle, $\overline{CO} \cong \overline{AE}$, and $\measuredangle ABC = 130°$, find the measure of $\angle CDE$.

Solution: In order to determine $\measuredangle CDE$ we need to determine the measures of the component angles, $\angle EDA$ and $\angle CDA$. To find $\angle EDA$...

- B and C must lie on one side of the circle with E alone on the other because $\overline{AD}$ is a diameter.
- $\measuredangle AED = 90°$ because the inscribed angle is half of the associated central angle of 180°.
- Furthermore, $\measuredangle EDA = 30°$ because ΔAED is a right triangle with a hypotenuse twice the length of the short side, and is thus a $30° - 60° - 90°$ right triangle.

Now we can find the measure of $\angle CDA$.

- Since $\measuredangle ABC = 130°$, is an inscribed angle, its associated central angle is 260° so major arc $\overset{\frown}{\mathrm{AC}} = 260°$.
- Minor arc $\overset{\frown}{\mathrm{AC}}$, therefore, is 100°.
- $\measuredangle CDA$ must be 50° since it is an inscribed angle and is half the measure of the intercepted arc.

Together, $\measuredangle CDA + \measuredangle EDA = \boxed{80°}$.